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EQUATIONS OF ATMOSPHERIC OSCILLATIONS

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SUMMARY

It is shown that the neglect of the terms in the differential equations of atmospheric oscillations which are quadratic in the velocities is not justified in the case of free oscillations for heights greater than about 100 kilometers. This result is illustrated for two atmospheres of different temperature distributions where it is shown that at about 130 kilometers the neglected quadratic terms become equal to the retained linear terms and that they even exceed the latter at heights greater than 130 kilometers. A method is outlined for including the quadratic terms in the theory of atmospheric oscillations. The bearing of this result on atmospheric tides is discussed.

INTRODUCTION

In the theory of tides, whether in the ocean or in the atmosphere, it has been customary since the days of Laplace to neglect the terms in the differential equations which are quadratic in the velocities. This procedure is justified in the case of the classical theory of oceanic tides, since the tidal velocities are of the same order of magnitude throughout the depth of the ocean and their squares can be neglected. In the case of the tides in the atmosphere which are similar in structure to long waves propagated horizontally with vertical wave fronts, the particle velocities increase generally with height because of the decrease of density with height. At a sufficiently high level the particle velocity approaches the local velocity of sound, so that at such heights it is no longer permissible to neglect the squares of the velocities in comparison with their first powers. In the case of the atmospheric oscillation of largest amplitude, namely, the solar semi-diurnal tide, the particle velocity near the ground is about 35 centimeters per second, which is only 0.1 percent of the velocity of sound there. At 100 kilometers, however, the amplitude of the particle velocity, as deduced from the linearized theory, is greater than at the ground by a factor of 100 or more, while the local sound velocity remains of the same order of magnitude. As long as the quadratic terms, as

computed from the linearized theory, turn out to be negligible in comparison with the linear terms, the linearization can be justified a posteriori. When, however, the quadratic terms turn out to be comparable with the linear terms, the linearized theory no longer applies.

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SYMBOLS

g	acceleration due to gravity
H	depth in a uniform ocean
H_0	height of homogeneous atmosphere
k	wave number
p	pressure
p_0	unperturbed pressure at any level
p_1	amplitude of pressure variation caused by wave
R	gas constant
T_0	temperature at any level
t	time
u, w	components of velocity
V	phase velocity of wave (σ/k)
x, z	distances in x- and z-directions
γ	ratio of specific heats
$\lambda = \frac{\gamma - 1}{\gamma H_0} = \frac{2}{7 H_0}$	
ρ	density of air

ρ_0	unperturbed density
σ	frequency of wave
Ω	gravitational potential of tide-producing body

ORDER OF MAGNITUDE OF QUADRATIC TERMS

In this section the order of magnitude of the quadratic terms, which are neglected in the linearized theory, will be determined. Consider a long wave propagated in the x-direction with a vertical wave front, and assume a time factor $e^{i(\sigma t - kx)}$, where σ denotes the frequency and k , the wave number. For the purpose of this discussion the earth's rotation may be neglected and the differential equation of the principal component of velocity u may be written as

$$\begin{aligned} \rho \frac{du}{dt} &= \rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \right) \\ &= -\frac{\partial p}{\partial x} - \rho \frac{\partial \Omega}{\partial x} \end{aligned} \quad (1)$$

where w denotes the vertical component of velocity, ρ , the density, p , the pressure, and Ω , the gravitational potential of the tide-producing body. In the linearized tidal theory one substitutes $\rho_0 \partial u / \partial t$ for the term $\rho du/dt$, where ρ_0 denotes the unperturbed density. By this substitution equation (1) becomes linear in u . In order to estimate the order of magnitude of the error committed thereby, it will suffice to determine whether the term $u \partial u / \partial x$ is negligible in comparison with $\partial u / \partial t$. Now

$$\left. \begin{aligned} \frac{\partial u}{\partial t} &= i\sigma u \\ \frac{\partial u}{\partial x} &= -iku \\ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} &= i\sigma u [1 - (k/\sigma)u] \\ &= i\sigma u [1 - (u/V)] \end{aligned} \right\} \quad (2)$$

where V denotes the phase velocity of the wave (i.e., σ/k). The ratio of the neglected quadratic terms in the differential equations to the retained linear terms is therefore of the order of the ratio of the particle velocity to the phase velocity of the wave. The phase velocity is comparable with the mean sound velocity in the vertical column of the atmosphere in which most of the wave energy is concentrated. Now it follows from the linearized equation (1) that

$$p_1 \equiv p - p_0 = \rho_0 uV \quad (3)$$

$$\begin{aligned} \frac{u}{V} &= \frac{p_1}{\rho_0 V^2} = \frac{p_1(z)RT_0(z)}{p_0(z)V^2} \\ &= \frac{p_1(z)H_0(z)}{p_0(z)H} \end{aligned} \quad (4)$$

with

$$\left. \begin{aligned} H_0(z) &= RT_0(z)/g \\ V^2 &\equiv gH \end{aligned} \right\} \quad (5)$$

The factor $H_0(z)/H$ varies relatively little with height, so that u/V is essentially determined by the variation of $p_1(z)/p_0(z)$, which is the ratio of the amplitude of the pressure variation caused by the wave to the unperturbed pressure at any level.

The magnitude of u/V in equation (4) can be exhibited most easily for the case of the free oscillation of an atmosphere of constant temperature. Here

$$\left. \begin{aligned} \rho_0(z) &= \rho_0(0) \exp(-z/H_0) \\ p_0(z) &= p_0(0) \exp(-z/H_0) \end{aligned} \right\} \quad (6)$$

$$\left. \begin{aligned} H &= \gamma H_0 \\ p_1 &= p_1(0) \exp[\lambda z + i(\sigma t - kx)] \end{aligned} \right\} \quad (7)$$

$$\lambda = \frac{\gamma - 1}{\gamma H_0} = \frac{2}{7H_0} \quad (8)$$

$$\left| \frac{u}{V} \right| = \frac{p_1'(0)}{\gamma p_0(0)} \exp(9z/7H_0) \\ \approx \frac{1}{1140} \exp(z/5.6) \quad (9)$$

where z is expressed in kilometers, and the values $H = 7.87$ kilometers and $p_1(0) = 1$ millimeter have been assumed to correspond to resonance with the solar semidiurnal tide. The right-hand side of equation (9) approaches unity at 40 kilometers, so that above 40 kilometers the neglected terms in the differential equations of tidal motion are larger than the linear terms, which have been retained.

Before proceeding to a discussion of this result, the results will be given here of similar determinations of $|u/V|$ for two atmospheres in which the temperature is not assumed to be constant with height but to have a distribution more closely in agreement with observations.

Figure 1 gives the ratio $\left| u \frac{\partial u / \partial x}{\partial u / \partial t} \right|$ for an atmosphere having the temperature distribution shown by curve T. This is the atmosphere B treated by Pekeris in reference 1, which was found to possess a resonance period close to 12 solar hours, with $H = 8.19$ kilometers. It is seen that the quadratic terms are less than 10 percent of the linear terms in the first 80 kilometers of height. At 100 kilometers this ratio reaches a value of 28 percent, to become greater than unity above about 125 kilometers. Figure 2 gives similar results for an atmosphere treated by Weekes and Wilkes (reference 2), which was also found to resonate with a period close to 12 solar hours. Here, the quadratic terms are less than 11 percent of the linear terms in the first 100 kilometers of height but exceed the linear terms at elevations greater than about 130 kilometers.

DISCUSSION OF RESULTS

In the previous section it was shown that the linearized theory of atmospheric oscillations ceases to be valid at elevations greater than 80 to 100 kilometers, because at those heights it is no longer permissible to neglect in the differential equations the terms which are quadratic in the velocities. The inclusion of these quadratic terms in the

theory is a task of much greater complexity than the solution of the original linearized equations. A procedure which could be followed in obtaining a solution of the complete differential equations is to expand the exact solution into a series proceeding in powers of the amplitude, starting with the solution of the linearized equations as a first approximation. Such an expansion would be expected to converge rapidly below about 100 kilometers. At higher elevations the exact solution would be expected to differ materially from that represented by the first approximation.¹

Pending the development of the exact solution for atmospheres of given temperature distributions, a brief qualitative discussion will now be given of the possible effects of the neglected quadratic terms in the tidal equations on the resonance theory of atmospheric tides.

Higher harmonics.— The solution of the exact wave equations for the free oscillations of the atmosphere will be periodic in time with an admixture of higher harmonics whose relative magnitude increases with elevation. A sinusoidal tidal potential will excite in the lower atmosphere a nearly pure sinusoidal oscillation of the same period, but at high elevations the oscillation will contain, besides the fundamental, also strong higher harmonics. The amplitude of the fundamental in the E layer may differ considerably from that predicted by the linearized solution, because the quadratic terms in the equations also contribute to the fundamental.

Free period.— The quadratic terms will cause a change in the free period of oscillation of the atmosphere or, more directly, in the velocity of propagation of long waves. This change would, however, be expected to be small in those atmospheres in which the linearized solution shows that only a small fraction of the wave energy in an atmospheric column is contained in the upper layers where the quadratic terms become significant. In the example of an atmosphere of constant temperature treated in the preceding section, the quadratic terms become dominant above 40 kilometers. The part of the atmosphere above 40 kilometers carries, however, only 10 percent of the total wave energy in the atmospheric column, so that it is not likely that the inclusion of the quadratic terms will appreciably alter the period computed from the linear theory. This would also hold for the atmosphere (B) of figure 1, since it is seen that the area under

¹Some exploratory work on this method of solution has been made, showing that on account of the earth's rotation the second harmonic will not be amplified by resonance, as is the first harmonic. A contrary result would have proved a serious difficulty in the proposed method.

the E curve (energy density) above the 80-kilometer level is a small fraction of the total area under this curve. On the other hand, if the temperature distribution in the atmosphere is such that a substantial portion of the wave energy in a column is located at the high levels where the quadratic terms are significant, then the free period of such an atmosphere may differ radically from that computed on the basis of linear theory. An atmosphere of the latter type is one having a positive temperature lapse rate in the outer layers.

Resonance theory.- It would follow from the preceding discussion that, provided the temperature distribution in the outer layer of the atmosphere is of the type shown in figure 1, the linear theory yields a good approximation to the amplification of the tide and the free period. Thus the claims of the resonance theory are not weakened for this type of atmosphere by consideration of the effects of the quadratic terms. On the other hand, the linear theory cannot be relied upon for a description of the features of the tidal oscillations in and above the E layer.

Effects of temperature distribution above 100 kilometers.- Since in the E layer and higher elevations the quadratic terms in the tidal equations become dominant, it is clear that any consequences, drawn on the basis of the linear theory, which result from assumed specific changes in the temperature distribution in and above the E layer are open to question. Thus the explanation given by Weekes and Wilkes (reference 2) of the observations on the lunar tide in the E layer made by Appleton and Weekes (reference 3) cannot be accepted, because it is dependent on the existence of a temperature maximum in the E layer where the quadratic terms turn out to be not only not negligible but twice as large as the retained linear terms.

The question as to whether the resonance theory of atmospheric tides is reconcilable with an assumed positive temperature lapse rate in the outer atmospheric layers must await a detailed attack on the solution of the nonlinearized tidal equations.

Effect of density distribution on atmospheric oscillations.- The emergence of the quadratic terms in the tidal equations is but a concealed effect of the density on the atmospheric oscillations, since the increase with height of the amplitude of the particle velocities is due to the decrease with height of the density.

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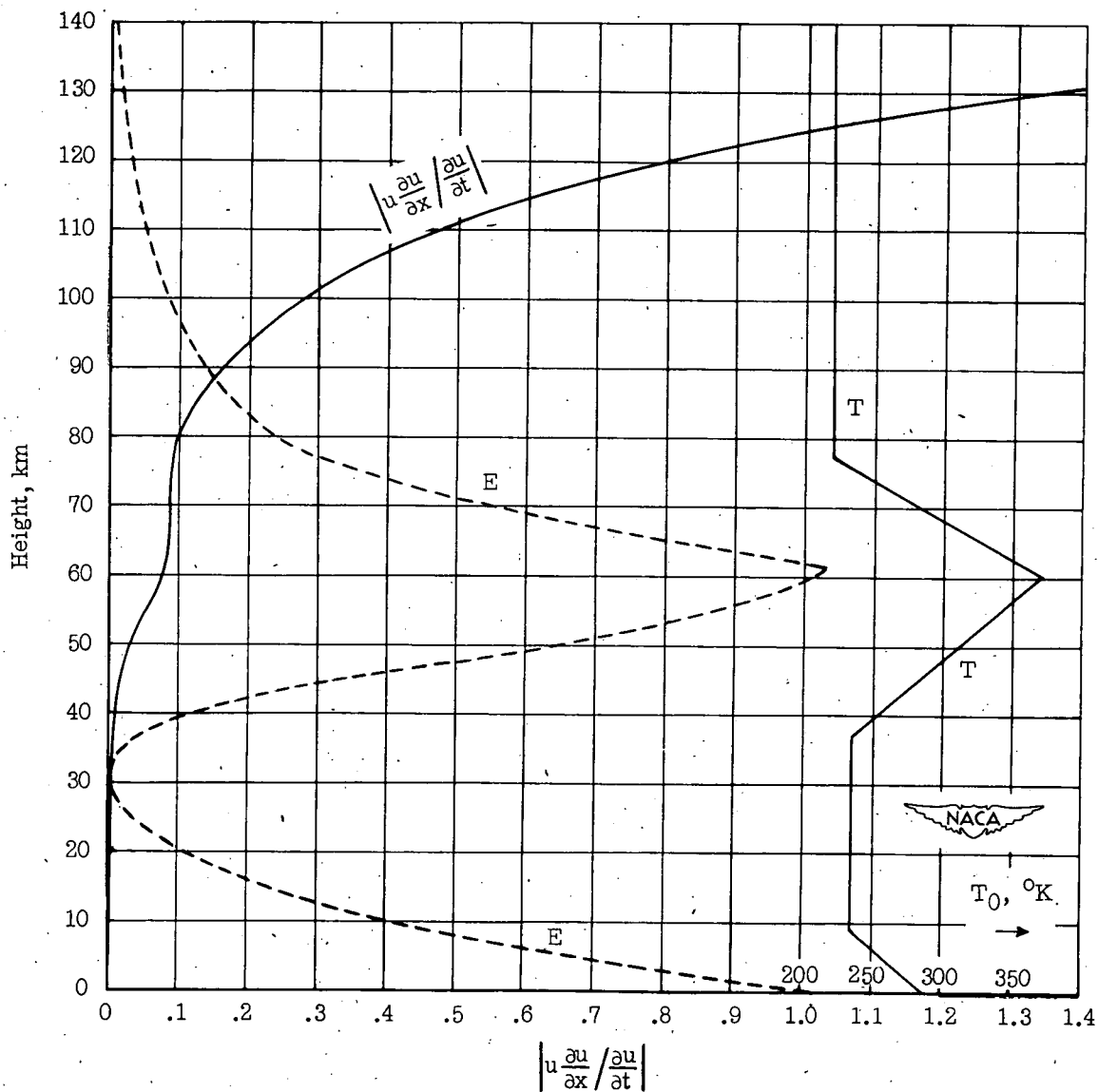


Figure 1.- Ratio of neglected term $u(\partial u/\partial x)$ to term $\partial u/\partial t$ which is retained in equations of atmospheric oscillations. Curve T is assumed temperature distribution; E, the wave-energy density.

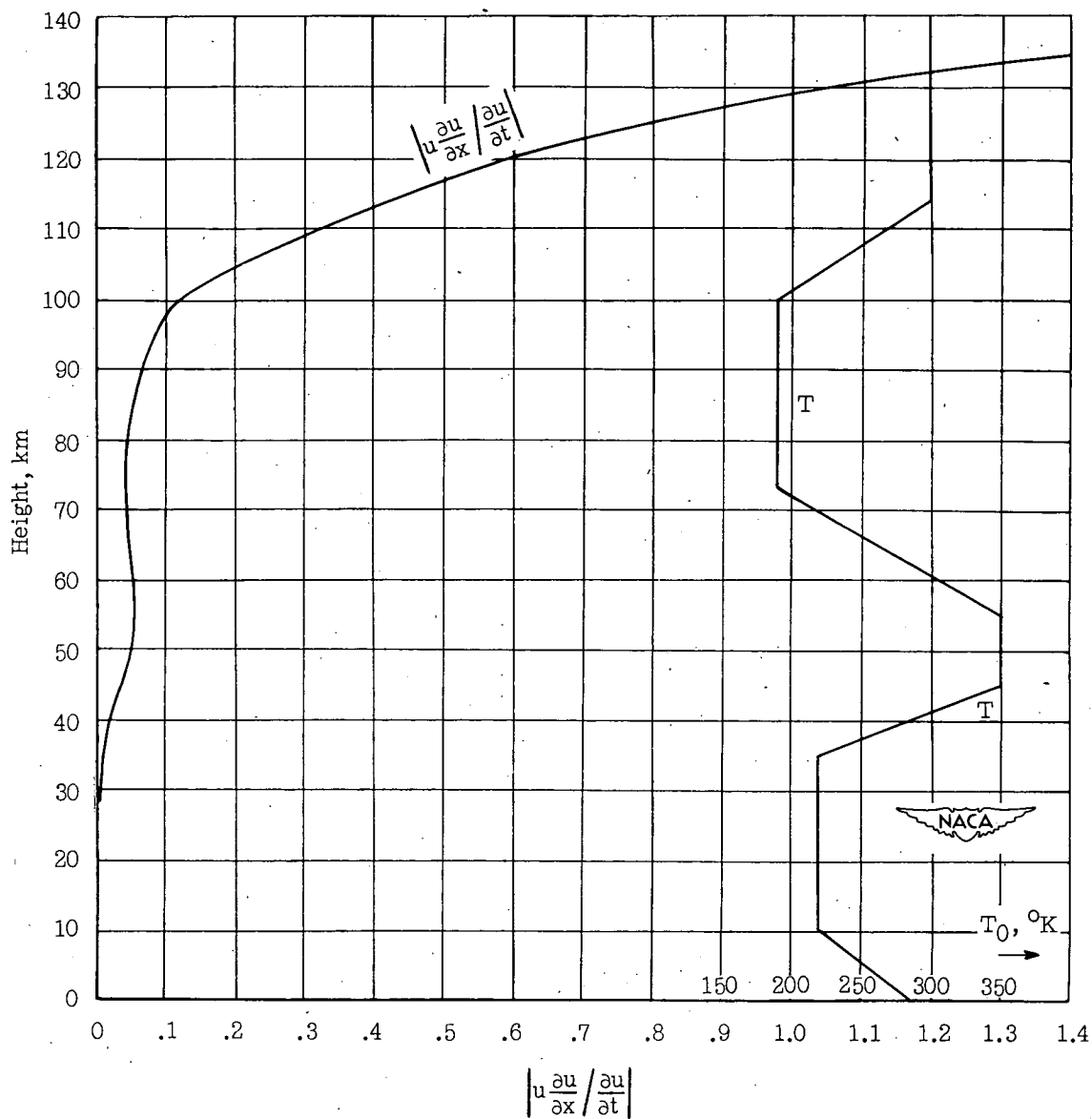


Figure 2.- Ratio of neglected term $u(\partial u/\partial x)$ to term $\partial u/\partial t$ which is retained in theory of atmospheric oscillations. Curve T shows assumed temperature distribution.